UK INTERMEDIATE MATHEMATICAL CHALLENGE

THURSDAY 2nd FEBRUARY 2012

Organised by the **United Kingdom Mathematics Trust**
from the School of Mathematics, University of Leeds

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The Actuarial Profession
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SOLUTIONS LEAFLET

This solutions leaflet for the IMC is sent in the hope that it might provide all concerned with some alternative solutions to the ones they have obtained. It is not intended to be definitive. The organisers would be very pleased to receive alternatives created by candidates.

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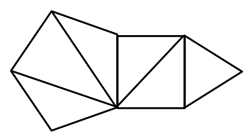
1. **B** 3 is the only one of the four numbers which is prime. The sums of the digits of the other three numbers are 6, 9, 12 respectively. These are all multiples of 3, so 33, 333, 3333 are all multiples of 3.

2. **D** The following triples of positive integers all sum to 7:

(1, 1, 5), (1, 2, 4), (1, 3, 3); (2, 2, 3).

In only one of these are the three integers all different, so the required integers are 1, 2, 4 and their product is 8.

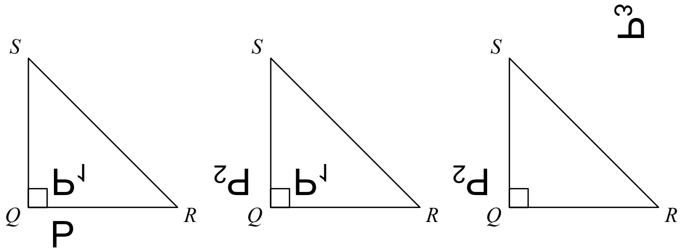
3. **E** The diagram shows that the interior angles of the polygon may be divided up to form the interior angles of six triangles. So their sum is $6 \times 180°$.



4. C The digits to be used must be 9, 8, 7, 6. If any of these were to be replaced by a smaller digit, then the sum of the two two-digit numbers would be reduced. For this sum to be as large as possible, 9 and 8 must appear in the 'tens' column rather than the 'units' column. So the largest possible sum is 97 + 86 or 96 + 87. In both cases the sum is 183.

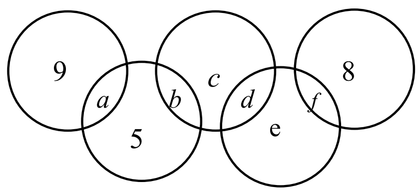
5. D The difference between the two given times is 24 hours 50 minutes $= (24 \times 60 + 50)$ minutes $= (1440 + 50)$ minutes $= 1490$ minutes.

6. A The diagram shows the result of the successive reflections.



7. C The primes in question are 5 and 11. The only primes smaller than 5 are 2 and 3. However neither 8 nor 9 is prime so p and q cannot be 2 and 8, nor 3 and 9.

8. A Referring to the diagram, $a = 11 - 9 = 2$; $b = 11 - 5 - a = 4$; $f = 11 - 8 = 3$. So the values of c, d and e are 1, 6, 7 in some order. If $c = 1$ then $d = 6$, but then e would need to be 2, not 7.



If $c = 6$, then $d = 1$ and $e = 7$ and this is a valid solution. Finally, if $c = 7$ then d would need to equal 0, which is not possible. So in the only possible solution, * is replaced by 6.

9. B 1% of 1 000 000 = 1 000 000 ÷ 100 = 10 000. So the least number of fleas which will be eradicated is 1 000 000 − 10 000 = 990 000.

10. D The table shows the first 12 positive integers, N, and the sum, S, of the factors of N excluding N itself. As can be seen, 12 is the first value of N for which this sum exceeds N, so 12 is the smallest abundant number.

N	1	2	3	4	5	6	7	8	9	10	11	12
S	0	1	1	3	1	6	1	7	4	8	1	16

(*Note that for N* = 6 *the sum also equals* 6. *For this reason,* 6 *is known as a 'perfect number'. After* 6, *the next two perfect numbers are* 28 *and* 496.)

11. C Opposite angles of a parallelogram are equal so $\angle QPS = 50°$.
Therefore $\angle QPT = 112°$ and, as triangle QPT is isosceles,
$\angle PQT = (180° - 112°) \div 2 = 34°$.
As $PQRS$ is a parallelogram, $\angle PQR = 180° - 50° = 130°$.
So $\angle TQR = 130° - 34° = 96°$.

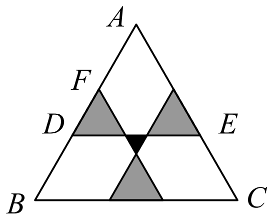
12. B The values of the expressions are £5.40, £6.00, £5.40, £5.40, £5.40 respectively.

13. D In the rally, approximately 90 shots were hit per minute for a total of 132 minutes. As $90 \times 130 = 11\,700$, D is the best alternative.

14. A The mean of the first three numbers is $\frac{1}{3}(20 + x)$; the mean of the last four numbers is $\frac{1}{4}(33 + x)$. Therefore $4(20 + x) = 3(33 + x)$, that is $80 + 4x = 99 + 3x$, so $x = 99 - 80 = 19$.

15. E $\frac{1}{2}+\frac{1}{3}\times\frac{1}{4}=\frac{1}{2}+\frac{1}{12}=\frac{7}{12}$; $\frac{1}{2}+\frac{1}{3}\div\frac{1}{4}=\frac{1}{2}+\frac{1}{3}\times\frac{4}{1}=\frac{1}{2}+\frac{4}{3}=\frac{11}{6}$; $\frac{1}{2}\times\frac{1}{3}\div\frac{1}{4}=\frac{1}{2}\times\frac{1}{3}\times\frac{4}{1}=\frac{2}{3}$; $\frac{1}{2}-\frac{1}{3}\div\frac{1}{4}=\frac{1}{2}-\frac{1}{3}\times\frac{4}{1}=\frac{1}{2}-\frac{4}{3}=-\frac{5}{6}$; $\frac{1}{2}-\frac{1}{3}\times\frac{1}{4}=\frac{1}{2}-\frac{1}{12}=\frac{5}{12}$.
Of the fractions $\frac{7}{12}, \frac{11}{6}, \frac{2}{3}, -\frac{5}{6}, \frac{5}{12}$, the closest to 0 is $\frac{5}{12}$.

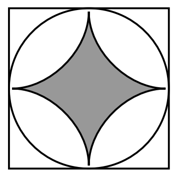
16. B As triangle ABC is equilateral, $\angle BAC = 60°$. From the symmetry of the figure, we may deduce that $AD = DE$ so triangle ADE is equilateral. The length of the side of this equilateral triangle = length of $DE = (5 + 2 + 5)$ cm =12 cm. So $AF = AD - AF = (12 - 5)$ cm $= 7$ cm. By a similar argument, we deduce that $BD = 7$ cm, so the length of the side of triangle $ABC = (7 + 5 + 7)$ cm $= 19$ cm.



17. E The terms of the sequence are 6, 3, 14, 7, 34, 17, 84, 42, 21, 104, 52, 26, 13, 64, 32, 16, 8, 4, 2, 1, 4, 2, 1, … . As can be seen, there will now be no other terms in the sequence other than 4, 2 and 1. It can also be seen that the only values of n for which the n th term = n are 13 and 16.

18. C After traversing the first semicircle, Peri will be at the point (8, 0); after the second semicircle Peri will be at (4, 0) and after the third semicircle, Peri will be at the point (2, 0).

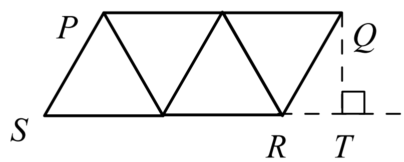
19. A The diagram shows the original diagram enclosed within a square of side $2r$, where r is the radius of the original circle. The unshaded area of the square consists of four quadrants (quarter circles) of radius r. So the shaded area is $4r^2 - \pi r^2 = r^2(4 - \pi)$. Therefore the required fraction is



$$\frac{r^2(4 - \pi)}{\pi r^2} = \frac{4 - \pi}{\pi} = \frac{4}{\pi} - 1.$$

20. D Let the sides of the rectangle, in cm, be $4x$ and $5x$ respectively.
Then the area of the square is $4x \times 5x\ \text{cm}^2 = 20x^2\ \text{cm}^2$. So $20x^2 = 125$, that is $x^2 = \frac{25}{4}$. Therefore $x = \pm\frac{5}{2}$, but x cannot be negative so the sides of the rectangle are 10 cm and 12.5 cm. Hence the rectangle has perimeter 45 cm.

21. A In the diagram, T is the foot of the perpendicular from Q to SR produced. Angles PQR and QRT are alternate angles between parallel lines so $\angle QRT = 60°$. Triangle QRT has interior angles of $90°$, $60°$, $30°$ so it may be thought of as being half of an equilateral triangle of side 1 unit, since the length of QR is 1 unit. So the lengths of RT and QT are $\frac{1}{2}$ and $\frac{\sqrt{3}}{2}$ units respectively.



Applying Pythagoras' Theorem to ΔQST, $SQ^2 = ST^2 + QT^2 = \left(\frac{5}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2 = \frac{25}{4} + \frac{3}{4} = 7$.
So the length of SQ is $\sqrt{7}$ units.

22. E The median number of cups of coffee is the median of a sequence of 190 positive integers $(t_1, t_2, \ldots, t_{190})$. Let the sum of these terms be S.
The median of the 190 numbers is $\frac{1}{2}(t_{95} + t_{96})$. The alternatives imply that the median cannot be greater than 3.5. The next lowest possible value for the median would be 4. For this to be possible, $t_{95} + t_{96} = 8$.
If $t_{95} = t_{96} = 4$ then the minimum value for S would occur if all other values of t were as small as possible, that is the first 94 values would all equal 1 and the last 94 values would all equal 4. In this case, $S = 94 \times 1 + 2 \times 4 + 94 \times 4 = 478$, whereas we are told that 477 cups of coffee were sold. Any other values of t_{95} and t_{96} such that $t_{95} + t_{96} = 8$ would produce a larger minimum value of S. For example, if $t_{95} = 3$ and $t_{96} = 5$ then the minimum value of S would be $94 \times 1 + 3 + 5 + 94 \times 5$, that is 572. So the median of the 190 terms cannot be 4, but it is possible for it to be 3.5. If the first 94 terms all equal 1, $t_{95} = 3$ and $t_{96} = 4$ and the last 94 terms all equal 4 then $S = 477$ as required and the median is $\frac{1}{2}(3 + 4) = 3.5$.
So the maximum possible value of the median number of cups of coffee bought per customer is 3.5.

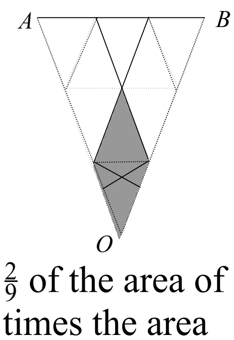
23. C As in the solution for Q21, ΔPTS may be thought of as half an equilateral triangle, so TS has length 1 unit. Therefore ΔSRT is isosceles and, as $\angle TSR = 120°$, $\angle SRT = \angle STR = 30°$. So $\angle TRQ = 45° - 30° = 15°$. Using the exterior angle theorem in ΔTQR, $\angle TQR = \angle STR - \angle TRQ = 30° - 15° = 15°$. So ΔTQR is isosceles with $TQ = TR$. However, ΔPRT is also isosceles with $PT = TR$ since $\angle PRT = \angle TPR = 30°$. Therefore $TQ = TP$, from which we deduce that PQT is an isosceles right-angled triangle in which $\angle PQT = \angle QPT = 45°$. So $\angle QPR = \angle QPT + \angle TPS = 45° + 30° = 75°$.

24. D The nature of the spiral means that 4 is in the top left-hand corner of a 2×2 square of cells, 9 is in the bottom right-hand corner of a 3×3 square of cells, 16 is in the top left-hand corner of a 4×4 square of cells and so on. To find the position of 2012 in the grid, we note that $45^2 = 2025$ so 2025 is in the bottom right-hand corner of a 45×45 square of cells and note also that $47^2 = 2209$. The table below shows the part of the grid in which 2012 lies. The top row shows the last 15 cells in the bottom row of a 45×45 square of cells, whilst below it are the last 16 cells in the bottom row of a 47×47 square of cells.

2011	2012	2013	2014	2015	2016	2017	2018	2019	2020	2021	2022	2023	2024	2025	
2194	2195	2196	2197	2198	2199	2200	2201	2202	2203	2204	2205	2206	2207	2208	2209

So 2195 lies below 2012.

25. B The diagram shows part of the ceramic.



A and B are vertices of the outer octagon, which has O at its centre. The solid lines are part of the original figure, whilst the broken lines OA, OB, two broken lines which are parallel to AB and broken lines parallel to OA and OB respectively have been added. As can be seen, these lines divide ΔOAB into nine congruent triangles. The shaded portion of triangle has area equal to that of two of the triangles. So $\frac{2}{9}$ of the area of ΔOAB has been shaded. Now the area of the outer octagon is eight times the area of ΔOAB and the area of shaded portion of the design is eight times the area of the shaded portion of ΔOAB so the fraction of the octagon which is shaded is also $\frac{2}{9}$.